Journal of Physics:
Conference Series

PURPOSE-LED PUBLISHING™

PAPER • OPEN ACCESS

Positive integer solutions of an equation involving Euler, generalized Euler, and Smarandache functions

To cite this article: Lianxia Jiang *et al* 2025 *J. Phys.: Conf. Ser.* **2964** 012044

View the article online for updates and enhancements.

You may also like

- The algebraic structure on the neutrosophic triplet set
S Suryoto, Harjito and T Udjiani
- e-open Sets in N_{me}-Topological Spaces
A. Vadivel and P. Thangaraja
- Q-Smarandache Fuzzy Implicative Ideal of QSmarandache BH-Algebra
Husein Hadi Abbass and Qasim Mohsin Luhaib

This content was downloaded from IP address 72.211.185.17 on 03/03/2025 at 15:37

Positive integer solutions of an equation involving Euler, generalized Euler, and Smarandache functions

Lianxia Jiang*, Li Wang, Sibao Zhang and Jiangang Tang

School of Mathematics and Statistics, Kashi University, Kashi, 844000, China

*Corresponding author's email: jianglianxia0908@sina.com

Abstract: This study investigates the positive integer solutions of an equation involving the Smarandache function. The equation is given by $t\varphi(n)=\varphi_2(n)+S(n^{16})$, where $\varphi(n)$ represents the Euler function, $\varphi_e(n)$ represents the generalized Euler function with e as a positive integer, and $S(n)$ represents the Smarandache function. The solutions of this equation are discussed, and it is proven that the equation only has positive integer solutions when $t=1,2,3,6,7,9,12,17,18,19$. Furthermore, all positive integer solutions of the equation are provided.

1. Introduction

We make n be a positive integer. The Euler function $\varphi(n)$ is defined as the number of positive integers that are coprime with n among the numbers $1,2,\cdots,n$. The generalized Euler function $\varphi_e(n)$ is defined as the number of positive integers that are coprime with n among the numbers $1,2,\cdots,\left[\frac{n}{e}\right]$, where e is a positive integer and $\left[\frac{*}{*}\right]$ denotes the floor function. The Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n\mid m!$. The study of number-theoretic functions and their relation to the solvability of related indeterminate equations has long been a topic of interest in number theory [1]. Previous works have discussed the solvability of equations involving the Euler function $\varphi(n)$ [2-4], the generalized Euler function $\varphi_2(n)$ [5], and the Smarandache function $S(n)$ [6-10]. There is also research on the solvability of equations that combine the Euler function $\varphi(n)$, the generalized Euler function $\varphi_e(n)$, and the Smarandache function $S(n)$. For example, Zhang et al. [11] discusse the solvability of the equation $K\varphi(Y)=\varphi_2(Y)+S(Y^8)$, Jiang et al. [12] discuss the solvability of two equations in the form of $K\varphi(n)=\varphi_2(n)+S(n^m)$, and Zheng [13] discusses the solvability of the equation $m\varphi(n)=\varphi_2(n)+S(n^{10})$. In this paper, we will discuss the solvability of the equation $t\varphi(n)=M\varphi_2(n)+NS(n^L)$ when $L=16$, $M=N=1$,

$$t\varphi(n)=\varphi_2(n)+S(n^{16}) \tag{1}$$

and provide all positive integer solutions to this equation.

Content from this work may be used under the terms of the Creative Commons Attribution 4.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd

2. Related lemmas

Lemma 2.1. If $n=\prod_{i=1}^{k} q_i^{\alpha_i}$, then $S(n)=\max\left(S\left(q_1^{\alpha_1}\right),S\left(q_2^{\alpha_2}\right),\cdots,S\left(q_k^{\alpha_k}\right)\right)$.

Lemma 2.2. [9] For a prime number p and a positive integer k, we have $S\left(p^k\right)\le kp$; in particular, when $k<p$, we have $S\left(p^k\right)=kp$.

Lemma 2.3. [14] For $n\ge 3$, we have $\varphi_2(n)=\dfrac{\varphi(n)}{2}$.

Lemma 2.4. [15] For $n\ge 3$, $\varphi(n)$ is an even number.

3. Main theorem and proof

Theorem 3.1. The indeterminate Equation (1) only has positive integer solutions when $t=1,2,3,6,7,9,12,17,18,19$, and

(1) When $t=1$, Equation (1) has positive integer solutions $n=1$, 847, 972, 1000, 1029, 1089, 1372, 1500, 1694,
2058, 2178;

(2) When $t=2$, Equation (1) has a positive integer solution $n=363,484,726$;

(3) When $t=3$, Equation (1) has a positive integer solution $n=81,162,169,338$;

(4) When $t=6$, Equation (1) has a positive integer solution $n=27,54$;

(5) When $t=7$, Equation (1) has a positive integer solution $n=24,25,50$;

(6) When $t=9$, Equation (1) has a positive integer solution $n=51,68,102$;

(7) When $t=12$, Equation (1) has a positive integer solution $n=9,18$;

(8) When $t=17$, Equation (1) has a positive integer solution $n=11,22$;

(9) When $t=18$, Equation (1) has a positive integer solution $n=5,10$;

(10) When $t=19$, Equation (1) has a positive integer solution $n=2$.

Proof. According to the definition of the generalized Euler function $\varphi_e(n)$, we have $\varphi_2(1)=0$ and $\varphi_2(2)=1$. For Equation (1), when $n=1$, we have $t\varphi(1)=\varphi_2(1)+S\left(1^{16}\right)$, which gives $t=1$. Therefore, when $t=1$, $n=1$ is a solution to Equation (1) (Here, "solution" refers to positive integer solutions, the same applies to the following). When $n=2$, we have $t\varphi(2)=\varphi_2(2)+S\left(2^{16}\right)$, which gives $t=1+18=19$. Therefore, when $t=19$, $n=2$ is a solution to Equation (1). For $n\ge 3$, we can assume that

$$n=\prod_{i=1}^{k} q_i^{\alpha_i}\ge 3$$

where $q_1<q_2<\cdots<q_k$ are prime numbers. According to Lemma 1, we have:

$$S(n)=\max\left(S\left(q_1^{\alpha_1}\right),S\left(q_2^{\alpha_2}\right),\cdots,S\left(q_k^{\alpha_k}\right)\right)=S\left(q^{\alpha}\right) \qquad (2)$$

where $n=q^{\alpha}n_1$, $(q,n_1)=1$. Therefore, we have:

$$\varphi(n)=\varphi\left(q^{\alpha}n_1\right)=\varphi(q^{\alpha})\varphi(n_1)=q^{\alpha-1}(q-1)\varphi(n_1) \qquad (3)$$

By Lemma 2.4, combined with Equations (2) and (3), Equation (1) can be transformed into:

$$(2t-1)q^{\alpha-1}(q-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{16\alpha}\right) \qquad (4)$$

Next, we will discuss Equation (4) by classifying and analyzing the cases based on different values of α.

Case 1. When $\alpha=1$, we have $(2t-1)(q-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{16}\right)$.

When $q=2$, for Equation (4), we have $(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(2^{16}\right)=36$. Therefore, we have $2t-1=1$, $\varphi(n_1)=36$, which gives $t=1$, $n_1=37,57,63,74,76,108,114,126$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=74,114,126$. However, after checking, none of $n=74,114,126$ is a solution to Equation (1). When $2t-1=3$, $\varphi(n_1)=12$, we have $t=2$, $n_1=13,21,26,28,36,42$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=26,42$. However, after checking, none of $n=26,42$ is a solution to Equation (1). When $2t-1=9$, $\varphi(n_1)=4$, we have $t=5$, $n_1=5,8,10,12$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=10$. However, after checking, $n=10$ is not a solution to Equation (1).

When $q=3$, for Equation (4), we have $(2t-1)\varphi(n_1)=S\left(n^{16}\right)=S\left(3^{16}\right)=36$. Therefore, we have $2t-1=1$, $\varphi(n_1)=36$, which gives $t=1$, $n_1=37,57,63,74,76,108,114,126$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=111,222,228$. However, after checking, none of $n=111,222,228$ is a solution to Equation (1). When $2t-1=3$, $\varphi(n_1)=12$, we have $t=2$, $n_1=13,21,26,28,36,42$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=39,78,84$. However, after checking, none of $n=39,78,84$ is a solution to Equation (1). When $2t-1=9$, $\varphi(n_1)=4$, we have $t=5$, $n_1=5,8,10,12$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=15,24,30$. However, after checking, none of $n=15,24,30$ is a solution to Equation (1).

When $q=5$, for Equation (4), we have $2(2t-1)\varphi(n_1)=S\left(n^{16}\right)=S\left(5^{16}\right)=70$, which means $(2t-1)\varphi(n_1)=35$. Therefore, we have $2t-1=35$, $\varphi(n_1)=1$, which gives $t=18$, $n_1=1,2$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=5,10$. After checking, we find that both $n=5$ and $n=10$ are solutions to Equation (1).

When $q=7$, for Equation (4), we have $3(2t-1)\varphi(n_1)=S\left(n^{16}\right)=S\left(7^{16}\right)=98$, which has no solution.

When $q=11$, for Equation (4), we have $5(2t-1)\varphi(n_1)=S\left(n^{16}\right)=S\left(11^{16}\right)=165$, which means $(2t-1)\varphi(n_1)=33$. Therefore, we have $2t-1=33$, $\varphi(n_1)=1$, which gives $t=17$, $n_1=1,2$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=11,22$. After checking, we find that both $n=11$ and $n=22$ are solutions to Equation (1).

When $q=13$, for Equation (4), we have $6(2t-1)\varphi(n_1)=S\left(n^{16}\right)=S\left(13^{16}\right)=195$, which has no solution.

When $q=17$, for Equation (4), we have $16(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(17^{16}\right)=2\times17\times16$, which means $(2t-1)\varphi(n_1)=34$. Therefore, we have $2t-1=1$, $\varphi(n_1)=34$, and $2t-1=17$, $\varphi(n_1)=2$. When $2t-1=1$, $\varphi(n_1)=34$, we have $t=1$, but since 34 is not an Euler quotient, there is no solution for $\varphi(n_1)=34$. When $2t-1=17$, $\varphi(n_1)=2$, we have $t=9$, $n_1=3,4,6$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=51,68,102$. After checking, we find that both $n=51$, $n=68$, and $n=102$ are solutions to Equation (1).

When $q=19$, for Equation (4), we have $18(2t-1)\varphi(n_1)=2S(n^{16})=2S(19^{16})=2\times19\times16$, which means $9(2t-1)\varphi(n_1)=304$. This equation has no solution.

When $q=23$, for Equation (4), we have $22(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(23^{16}\right)=2\times23\times16$, which means $11(2t-1)\varphi(n_1)=368$. This equation has no solution.

When $q=29$, for Equation (4), we have $28(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(29^{16}\right)=2\times29\times16$, which means $7(2t-1)\varphi(n_1)=232$. This equation has no solution.

When $q=31$, for Equation (4), we have $30(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(31^{16}\right)=2\times31\times16$, which means $15(2t-1)\varphi(n_1)=496$. This equation has no solution.

When $q\geq37$, for Equation (4), we have $(2t-1)(q-1)\varphi(n_1)=2S\left(q^{16}\right)=32q$, which means $\left[(2t-1)\varphi(n_1)-32\right](q-1)=32$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 2. When $\alpha=2$, we have $q(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{32})$.

When $q=2$, for Equation (4), we have $2(2t-1)\varphi(n_1)=2S(n^{16})=2S(2^{32})=2\times34$, which means $(2t-1)\varphi(n_1)=34$. Therefore, we have $2t-1=1$, $\varphi(n_1)=34$, and $2t-1=17$, $\varphi(n_1)=2$. When $2t-1=1$, $\varphi(n_1)=34$, we have $t=1$. However, since 34 is not an Euler quotient, there is no solution for $\varphi(n_1)=34$. When $2t-1=17$, $\varphi(n_1)=2$, we have $t=9$, $n_1=3,4,6$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=12$. After checking, we find that $n=12$ is not a solution to Equation (1).

When $q=3$, for Equation (4), we have $6(2t-1)\varphi(n_1)=2S(n^{16})=2S(3^{32})=2\times69$, which means $(2t-1)\varphi(n_1)=23$. Therefore, we have $2t-1=23$, $\varphi(n_1)=1$, and we have $t=12$, $n_1=1,2$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=9,18$. After checking, we find that both $n=9$ and $n=18$ are solutions to Equation (1).

When $q=5$, for Equation (4), we have $20(2t-1)\varphi(n_1)=2S(n^{16})=2S(5^{32})=2\times130$, which means $(2t-1)\varphi(n_1)=13$. Therefore, we have $2t-1=13$, $\varphi(n_1)=1$, and we have $t=7$, $n_1=1,2$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=25,50$. After checking, we find that both $n=25$ and $n=50$ are solutions to Equation (1).

When $q=7$, for Equation (4), we have $42(2t-1)\varphi(n_1)=2S(n^{16})=2S(7^{32})=2\times196$. This gives $3(2t-1)\varphi(n_1)=14$, which has no solution for Equation (1).

When $q=11$, for Equation (4), we have $110(2t-1)\varphi(n_1)=2S(n^{16})=2S(11^{32})=2\times330$, which means $(2t-1)\varphi(n_1)=6$. Therefore, we have $2t-1=1$, $\varphi(n_1)=6$, and $2t-1=3$, $\varphi(n_1)=2$. When $2t-1=1$, $\varphi(n_1)=6$, we have $t=1$, $n_1=7,9,14,18$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=847,1089,1694,2178$. After checking, we find that $n=847,1089,1694,2178$ is a solution to Equation (1). When $2t-1=3$, $\varphi(n_1)=2$, we have $t=2$, $n_1=3,4,6$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=363,484,726$. After checking, we find that $n=363,484,726$ is a solution to Equation (1).

When $q=13$, for Equation (4), we have $156(2t-1)\varphi(n_1)=2S(n^{16})=2S(13^{32})=2\times390$, which means $(2t-1)\varphi(n_1)=5$. Therefore, we have $2t-1=5$, $\varphi(n_1)=1$, and we have $t=3$, $n_1=1,2$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=169,338$. After checking, we find that both $n=169$ and $n=338$ are solutions to Equation (1).

When $q=17$, for Equation (4), we have $272(2t-1)\varphi(n_1)=2S(n^{16})=2S(17^{32})=2\times527$. This gives $8(2t-1)\varphi(n_1)=31$, which has no solution for Equation (1).

When $q=19$, for Equation (4), we have $342(2t-1)\varphi(n_1)=2S(n^{16})=2S(19^{32})=2\times589$. This gives $9(2t-1)\varphi(n_1)=31$, which has no solution for Equation (1).

When $q=23$, for Equation (4), we have $506(2t-1)\varphi(n_1)=2S(n^{16})=2S(23^{32})=2\times713$. This gives $11(2t-1)\varphi(n_1)=31$, which has no solution for Equation (1).

When $q=29$, for Equation (4), we have $812(2t-1)\varphi(n_1)=2S(n^{16})=2S(29^{32})=2\times899$. This gives $14(2t-1)\varphi(n_1)=31$, which has no solution for Equation (1).

When $q=31$, for Equation (4), we have $930(2t-1)\varphi(n_1)=2S(n^{16})=2S(31^{32})=2\times961$. This gives $15(2t-1)\varphi(n_1)=31$, which has no solution for Equation (1).

When $q \geq 37$, for Equation (4), we have $q(q-1)(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(q^{32}) = 2 \times 32q$, which means $(2t-1)\varphi(n_1)(q-1) = 64$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 3. When $\alpha = 3$, we have $q^2(q-1)(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(q^{48})$.

When $q = 2$, for Equation (4), we have $4(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(2^{48}) = 2 \times 52$, which means $(2t-1)\varphi(n_1) = 26$. Therefore, we have $2t-1 = 1$, $\varphi(n_1) = 26$, and $2t-1 = 13$, $\varphi(n_1) = 2$. When $2t-1 = 1$, $\varphi(n_1) = 26$, we have $t = 1$. However, since 26 is not an Euler quotient, there is no solution for $\varphi(n_1) = 26$. When $2t-1 = 13$, $\varphi(n_1) = 2$, we have $t = 7$, $n_1 = 3,4,6$. According to $n = q^{\alpha}n_1$, $(q,n_1) = 1$, we have $n = 24$. After checking, we find that $n = 24$ is a solution to Equation (1).

When $q = 3$, for Equation (4), we have $18(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(3^{48}) = 2 \times 99$, which means $(2t-1)\varphi(n_1) = 11$. Therefore, we have $2t-1 = 11$, $\varphi(n_1) = 1$, and we have $t = 6$, $n_1 = 1,2$. According to $n = q^{\alpha}n_1$, $(q,n_1) = 1$, we have $n = 27,54$. After checking, we find that both $n = 27$ and $n = 54$ are solutions to Equation (1).

When $q = 5$, for Equation (4), we have $100(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(5^{48}) = 2 \times 200$, which means $(2t-1)\varphi(n_1) = 4$. Therefore, we have $2t-1 = 1$, $\varphi(n_1) = 4$, and we have $t = 1$, $n_1 = 5,8,10,12$. According to $n = q^{\alpha}n_1$, $(q,n_1) = 1$, we have $n = 1000,1500$. After checking, we find that both $n = 1000$ and $n = 1500$ are solutions to Equation (1).

When $q = 7$, for Equation (4), we have $294(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(7^{48}) = 2 \times 294$, which means $(2t-1)\varphi(n_1) = 2$. Therefore, we have $2t-1 = 1$, $\varphi(n_1) = 2$, and we have $t = 1$, $n_1 = 3,4,6$. According to $n = q^{\alpha}n_1$, $(q,n_1) = 1$, we have $n = 1029,1372,2058$. After checking, we find that $n = 1029,1372,2058$ is a solution to to Equation (1).

When $q \geq 11$, for Equation (4), we have $q^2(q-1)(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(q^{48}) \leq 96q$, which means $q(q-1)(2t-1)\varphi(n_1) \leq 96$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 4. When $\alpha = 4$, we have $q^3(q-1)(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(q^{64})$.

When $q = 2$, for Equation (4), we have $8(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(2^{64}) = 2 \times 66$, which means $2(2t-1)\varphi(n_1) = 33$. This equation has no solution for Equation (1).

When $q = 3$, for Equation (4), we have $54(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(3^{64}) = 2 \times 135$, which means $(2t-1)\varphi(n_1) = 5$. Therefore, we have $2t-1 = 5$, $\varphi(n_1) = 1$, and we have $t = 3$, $n_1 = 1,2$. According to $n = q^{\alpha}n_1$, $(q,n_1) = 1$, we have $n = 81,162$. After checking, we find that both $n = 81$ and $n = 162$ are solutions to Equation (1).

When $q = 5$, for Equation (4), we have $500(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(5^{64}) = 2 \times 260$, which means $25(2t-1)\varphi(n_1) = 26$. This equation has no solution for Equation (1).

When $q \geq 7$, for Equation (4), we have $q^3(q-1)(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(q^{64}) \leq 128q$, which means $q^2(q-1)(2t-1)\varphi(n_1) \leq 128$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 5. When $\alpha = 5$, we have $q^4(q-1)(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(q^{80})$.

When $q = 2$, for Equation (4), we have $16(2t-1)\varphi(n_1) = 2S(n^{16}) = 2S(2^{80}) = 2 \times 84$, which means $2(2t-1)\varphi(n_1) = 21$. This equation has no solution for Equation (1).

When $q=3$, for Equation (4), we have $162(2t-1)\varphi(n_1)=2S(n^{16})=2S(3^{80})=2\times162$, which means $(2t-1)\varphi(n_1)=2$. Therefore, we have $2t-1=1$, $\varphi(n_1)=2$, and we have $t=1$, $n_1=3,4,6$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=972$. After checking, we find that $n=972$ is a solution to Equation (1).

When $q\geq5$, for Equation (4), we have $q^4(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{80})\leq160q$, which means $q^3(q-1)(2t-1)\varphi(n_1)\leq160$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 6. When $\alpha=6$, we have $q^5(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{96})$.

When $q=2$, for Equation (4), we have $32(2t-1)\varphi(n_1)=2S(n^{16})=2S(2^{96})=2\times100$, which means $4(2t-1)\varphi(n_1)=25$. This equation has no solution for Equation (1).

When $q=3$, for Equation (4), we have $486(2t-1)\varphi(n_1)=2S(n^{16})=2S(3^{96})=2\times198$, which means $27(2t-1)\varphi(n_1)=22$. This equation has no solution for Equation (1).

When $q\geq5$, for Equation (4), we have $q^5(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{96})\leq192q$, which means $q^4(q-1)(2t-1)\varphi(n_1)\leq192$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 7. When $\alpha=7$, we have $q^6(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{112})$.

When $q=2$, for Equation (4), we have $64(2t-1)\varphi(n_1)=2S(n^{16})=2S(2^{112})=2\times116$, which means $8(2t-1)\varphi(n_1)=29$. This equation has no solution for Equation (1).

When $q\geq3$, for Equation (4), we have $q^6(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{112})\leq224q$, which means $q^5(q-1)(2t-1)\varphi(n_1)\leq224$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 8. When $\alpha=8$, we have $q^7(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{128})$.

When $q=2$, for Equation (4), we have $128(2t-1)\varphi(n_1)=2S(n^{16})=2S(2^{128})=2\times130$, which means $32(2t-1)\varphi(n_1)=65$. This equation has no solution for Equation (1).

When $q\geq3$, for Equation (4), we have $q^7(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{128})\leq256q$, which means $q^6(q-1)(2t-1)\varphi(n_1)\leq256$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 9. When $\alpha=9$, we have $q^8(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{144})$.

When $q=2$, for Equation (4), we have $256(2t-1)\varphi(n_1)=2S(n^{16})=2S(2^{144})=2\times148$, which means $32(2t-1)\varphi(n_1)=37$. This equation has no solution for Equation (1).

When $q\geq3$, for Equation (4), we have $q^8(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{144})\leq288q$, which means $q^7(q-1)(2t-1)\varphi(n_1)\leq288$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 10. When $\alpha=10$, we have $q^9(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{160})$.

When $q=2$, for Equation (4), we have $512(2t-1)\varphi(n_1)=2S(n^{16})=2S(2^{160})=2\times164$, which means $64(2t-1)\varphi(n_1)=41$. This equation has no solution for Equation (1).

When $q\geq3$, for Equation (4), we have $q^9(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{160})\leq320q$, which means $q^8(q-1)(2t-1)\varphi(n_1)\leq320$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 11. When $\alpha \geq 11$, for $q \geq 2$, we have $q^{\alpha-1}(q-1)(2t-1)\varphi(n_1) = 2S\left(q^{16\alpha}\right) \leq 32\alpha q$, which means $q^{\alpha-2}(q-1)(2t-1)\varphi(n_1) \leq 32\alpha$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Combining the discussions in the 11 cases above, we can conclude Theorem 1. The proof is completed.

4. Conclusion

In this study, we investigated equations of the form tφ(n)=φ2(n)+S(n), where the Euler function φ(n), Smarandache function S(n), and generalized Euler function φe(n) are involved, with k and n belonging to the set of positive integers (Z+). By employing piecewise classification and combining the properties of these three functions with elementary methods, we determined the values of k that yield positive integer solutions for the corresponding equations when n=16, and provided all positive integer solutions for these equations. Furthermore, we demonstrated that the solvability problem of equations of the form tφ(n)=φ2(n)+S(n) for other positive integer values of n∈Z+ can also be addressed using the methodology described in this paper. This research contributes to the understanding of the relationships between the Smarandache function and the Euler function, as well as the generalized Euler function, in terms of their properties and behavior. Further investigations can be conducted to explore additional aspects and applications of these equations in number theory.

Acknowledgments

Research Fund of Basic Research Business Fees of Xinjiang Uygur Autonomous Region Universities (XJEDU2022P088); 2023 Kashi University "College Students Innovation and Entrepreneurship Training Program" project "Solubility Study of Indeterminate Equations with Generalized Euler Functions" interim results (2023018).

References

[1] HUA L G. Introduction to number theory [M]. Beijing: Science Press, 1979:13-14.
[2] ZHANG S B, YANG Y N, XI X Z. Several nonlinear equations on Euler function $\varphi(n)$ [J]. Journal of Henan University (Natural Science), 2019, 49(1):122-126.
[3] JIANG L X, ZHANG S B. The solvability of equation $\varphi(n) = 2^{\omega(n)}P^{\Omega(n)}$ [J]. Journal of Henan University (Natural Science), 2022, 52(2):247-252.
[4] DAI Y B, GAO L. Positive Integer solutions of nonlinear equations of ternary variable coefficients Euler functions [J]. Journal of Southwest China Normal University (Natural Science Edition), 2023, 48(4):32-36.
[5] ZHANG S B. Solutions of an equation involving two Generalized Euler function [J]. Journal of Jilin University (Science Edition), 2022, 60(2):189-201.
[6] ZHANG S B, JIANG L X. Positive integer solutions of arithmetic function equation $\varphi_4(X) = S\left(X^6\right)$ [J]. Journal of Inner Mongolia Agricultural University (Natural Science Edition), 2021, 42(6):108-111.
[7] ZHANG S B. The solvability of arithmetic function equation $\varphi_2(N) = S\left(N^{16}\right)$. Journal of Chongqing University of Technology (Natural Science), 2020, 34(7):249-254.
[8] ZHANG S B, JIANG L X. Discussion on the positive integer solution of equation $k\varphi(m) = S(m^t)$. Journal of Henan University (Natural Science), 2021, 51(3):367-372.
[9] YANG Z Y, ZHAO X Q, BAI J W. Solution of arithmetic function equation $\varphi_2(n) = S\left(n^{10}\right)$ [J]. Journal of Yanan University(Natural Science Edition), 2017, 36(1):9-12.
[10] XU H X, ZHAO X Q, ZHANG L X. Solution of arithmetic function equation $\varphi_2(n) = S\left(n^8\right)$ [J]. J. Jianghan Univ. (Nat. Sci. Ed.), 2016, 44(4):321-326.

[11] ZHANG S B, JIANG L X. The solutions of arithmetic function equation $K\varphi(Y)=\varphi_2(Y)+S(Y^8)$ [J]. Journal of Jiangxi Normal University (Natural Science), 2021, 45(2):194-197.
[12] JIANG L X, ZHANG S B, FU Y. Solvabilities of two equations of the form $K\varphi(n)=\varphi_2(n)+S(n^m)$ [J]. Journal of Capital Normal University (Natural Science Edition), 2021, 42(6):8-11, 70.
[13] ZHENG H. Positive integer of solutions to ternary Euler function equation $m\varphi(n)=\varphi_2(n)+S(n^{10})$ [J]. Jiangxi Science, 2022, 40(2):219-222.
[14] IANG S N, SHEN Z Y. The solution of the arithmetic functional equation $\Phi(n)=S(n^{11})$ and $\Phi_2(n)=S(n^{11})$ [J]. Journal of Zhejiang International Studies University, 2014(5):31-37.
[15] PAN C D, PAN C B. Elementary number theory [M]. Beijing: Peking University Press, 1992.